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# Rate Distortion Theory for Descriptive Statistics

Peter Harremoës

Niels Brock, Copenhagen Business College

Copenhagen, Denmark

E-mail: harremoës@iee.org

**Abstract**—Rate distortion theory was developed for optimizing lossy compression of data, but it also has a lot of applications in statistics. In this paper we will see how rate distortion theory can be used to analyze a complicated data set involving orientations of early Islamic mosques. The analysis involves testing, identification of outliers, choice of compression rate, calculation of optimal reconstruction points, and assigning “descriptive confidence regions” to the reconstruction points. In this paper the focus will be on the technical aspects, so the integrity of the data set and the interpretation of the results will not be discussed in this paper.

## I. INTRODUCTION

Rate distortion theory was introduced by Shannon as a tool for lossy compression. His goal was to compress an information source to a rate that could be sent through a communication channel with limited capacity. Applications of ideas from rate distortion theory for testing Goodness-of-Fit was studied in [1], [2]. The purpose of the present paper is to demonstrate that rate distortion theory can be used to solve a variety of statistical problems. The advantage of these methods is that they are descriptive in nature, which implies that we only need to make minimal assumptions about how the data were generated.

To illustrate our method we have analyzed data on the orientation of early mosques. A database with information on the orientation of 160 early mosques has been compiled by D. Gibson [3]. In this paper we will not discuss the integrity of the data in the database except that we have excluded the ancient mosque in Aqaba where it is unclear which wall was the qibla wall.

The data is analyzed using rate distortion theory and the calculations are done with the R program. There was a rate distortion package for the R program [4], but that package was not maintained. In addition we need extra features for the type of analysis explained in the present paper. For this reason we have developed a new package for R version 4.1.2 for solving rate distortion problems. The software has been developed as a general purpose package and incorporates the Blahut-Arimoto algorithm and the package is still in development but the present version is available online[5]. When fully developed and documented the package will be uploaded to CRAN. The specific data set has been analyzed using a R worksheet that can also be downloaded [6].

## II. HISTORICAL BACKGROUND

According to the Islamic traditions Islam was founded in 622 CE, but the history of early Islam was only written down several hundred years later. According to these traditions the

Muslims were praying facing Jerusalem until 624 CE where Prophet Muhammad received a revelation commanding him to face towards the Sacred Mosque (Mashid al-Ḥarām). Now-a-days Muslims pray facing Mecca in Saudi Arabia where Mashid al-Ḥarām is located. Inside the Sacred Mosque there is a building called the Ka’ba, which is at the center of the spiritual life of the Muslims.

The Qur’an (Q2:143-144) states that the direction of prayer (qibla) should be towards Mashid al-Ḥarām, but it is not mentioned where Mashid al-Ḥarām is located. The Ka’ba is mentioned as a destiny of pilgrimage, but it is not mentioned that the Ka’ba is within Mashid al-Ḥarām. A text by a Christian author Jacob of Edessa from around 700 CE states that the Arabs pray towards the Ka’ba, but it does not describe where the Ka’ba was located. Instead the text explains that the Arabs in Egypt face east, and the Arabs in Kufa pray facing west [7], [8]. This does not fit with a Ka’ba in Mecca. Grunebaum claims that there were a number of ka’bas at different locations in the Arabia [9, p. 24].

Early Muslim scholars were aware that many of the oldest mosques were not facing Mecca in Saudi Arabia. One theory is that the early Muslims did not know the exact direction towards Mecca. Given their ability to navigate through the desert this seems less likely [10]. There are a lot of indications that Islam has its origin in north western Arabia rather than in the area around Mecca [11], [12], and the earliest reference to Mecca outside the Qur’an date as late as 743 CE [13]. There is even an old theory that Petra in Jordan was the birthplace of Prophet Muhammad rather than Mecca [14].

We will use orientations of old mosques to provide information on the qiblas used during the formative years of Islam. Typically a mosque has a long qibla wall with a mihrab, a prayer niche, in the middle and the Muslims are praying facing the qibla wall. The earliest mosques did not have a mihrab, which makes the identification of early mosques a challenge. In these cases the identification of a qibla wall relies on other architectural features of the building as a whole. Many early mosques have a qibla that appear to be inconsistent with a direction towards Mecca in Saudi Arabia.

The Islamic traditions were written down during the Abbasid dynasty and before that dynasty came into power there had been several civil wars. In the written accounts the authors openly admit that they have been selective in their choice of narrative. Since the written accounts are late and biased towards the ruling Abbasids it is very difficult to judge which parts of these accounts are historically sound. Here we

will restrict the attention to the period before the Abbasid revolution in year 750 CE where this Abbasid dynasty came into power. We will subdivide the period in an early period from 622CE to the reforms of the Umayyad Caliph 'Abd al-Malik around 700 CE and a late period from 700 CE to the Abbaside revolution 750 CE.

### III. DISTORTION

In this paper *qibla* is a theory that assigns a certain bearing to each possible location of a mosque. We will compare the qibla bearing with the measured orientation of an ancient site. Both the qibla bearing and the orientation can be given as a number of degrees measured clockwise from geographical north. Normally the orientation will deviate from the qibla bearing that was intended by the people who built the mosque. Four main reasons for this are:

- 1) The architect may not have been able to determine the qibla bearing exactly.
- 2) Local obstacles or other practical problems may have influenced the orientation of the site.
- 3) The original structure may now be a ruin or it has been rebuilt so that the original orientation is difficult to determine.
- 4) Sometimes it is difficult to measure the orientation as discussed in [10].

We have to quantify how much the orientation  $o$  deviates from the qibla bearing  $b$  of the site. As distortion measure we use

$$\text{versin}(o - b) = 1 - \cos(o - b).$$

This is the standard method for measuring distortion (also called dispersion) in directional statistics [15, Sec. 2.3]. Circular variance can be translated to circular standard deviation using

$$\sigma = \frac{360^\circ}{2\pi} (-2 \ln(1 - V))^{1/2}.$$

The distribution with specified mean direction and specified circular standard deviation that maximize entropy is a von Mises distribution [16]. Thus, using  $\text{versin}(o - b)$  to calculate distortion correspond to using the von Mises distributions as our basic error model.

### IV. CALCULATION OF THE RATE DISTORTION FUNCTION

As source alphabet we use the sites given by their geographical coordinates and their bearing. As reconstruction points we use points given by their coordinates. The distortion of a site and a point is given by  $\text{versin}$  of the difference between the orientation of the site and bearing from the site to the reconstruction point. We note that the source alphabet is discrete but the reconstruction alphabet is continuous. we run the following algorithm in order to calculate the rate distortion function.

- 1) We create a number of random probability vectors over the source alphabet. These probability vectors are chosen according to a Dirichlet distribution.

- 2) For each probability vector over the source alphabet we calculate the optimal reconstruction point for these weights using the Nelder Mead algorithm.
- 3) With these reconstruction points we run the Blahut Arimoto algorithm [17], [18, Sec. 13.8] and get a coupling between source alphabet and reconstruction alphabet.
- 4) If one of the reconstruction points has probability close to zero, it is removed.
- 5) If the conditional distributions of source points given two different reconstruction points are close together then one of the reconstruction points is removed.
- 6) For each reconstruction point we replace the reconstruction point by a new reconstruction point that is optimal with respect to the joint distribution, and go back to step 3.

The algorithm stops when running 3. to 6. gives an improvement of the rate below a certain threshold. The Blahut Arimoto algorithm is iterated until the improvement is below a threshold that is 1/10 of the threshold used as stopping rule for iterating 3. to 6.

### V. TEST OF GREAT CIRCLES VS. RHUMB LINES

D. Gibson has compared the orientations of the mosques with the geodesic directions [10], but he has been criticized that calculations of great circles was not developed in the formative years of Islam [19], [20]. Aa method called the Indian Circle is described on old documents and corresponds to determination of bearings along rhumb lines [21]. Calculations based on great circles involve trigonometric formulas and astronomical observation that were developed by Muslim scholars later than the period that is the focus of this paper, but in principle a bearing may be determined by other means than calculations. Using bearings based on rhumb lines has the advantage that mosques that have the same orientation can be represented by a very distant reconstruction point.

The question is whether we should base the subsequent calculations on bearings calculated along great circles taking the curvature of earth into account or whether we should calculate bearings along rhumb lines corresponding to a flat earth. Formally, a statistical test is a binary decision based on data, so we should test of great circles versus rhumb lines. Our decision criteria is simple: we choose the model that gives the best compression.

On Figure 1 we compare the rate distortion curve when the bearings is based on great circles with the rate distortion curve based on bearings along rhumb lines. For the great circle distortion we have plotted a lower bound based on the tangents to the rate distortion curve. For the distortion based on the rhumb lines we have plotted an upper bound by plotting the cords between achievable rate distortion pairs.

As we see the rhumb line distortion gives smaller values of the distortion than the great circle distortion for rates less than approximately 0.3 nats. For greater rates the two curves are very close together. The reason that two distortion measures are so similar for high rates appear to be that the optimal

reconstruction points bifurcate and that the bifurcations tend to blur out the difference between the two distortion measures.

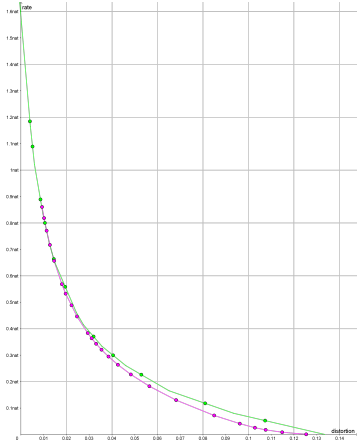


Fig. 1. The green polyline is a lower bound on the rate distortion function when we use bearings along great circles. The purple polyline is an upper bound on the rate distortion function when the bearings are calculated along rhumb lines. The dots are points where rate and distortion have been calculated.

Since the bearings based rhumb lines give a slightly better fit with data we will use bearings based on rhumb in the rest of this paper.

#### VI. OUTLIER DETECTION FOR SITES BEFORE 700 CE

We will parameterize the rate distortion curve by its slope  $s$ . For each slope  $s$  we get a list of reconstruction points and a joint distribution of sites and reconstruction points. Both the reconstruction points and the joint distribution will depend on the rate, but the individual reconstruction points are quite robust to changes in the rate. If we increase the rate the reconstruction point will split into a large number of reconstruction points each with little weight. If we decrease the rate then reconstruction points will merge together. It requires a little experimentation to find a rate that gives interesting result. We here will use the slope  $s = -83$ . Further justification of this value will be given in Section VII.

Here we will make a rate distortion analysis on the set of the 20 sites that are dated prior to year 700 CE. If we compress with slope  $s = -83$  we get a mean distortion of 0.00395 corresponding to a circular standard deviation of  $5.2^\circ$ . At this slope the rate is 0.4840 nats. We get the following reconstruction points.

Rec. point	Latitude	Longitude	Weight	Distortion
<b>Pe</b>	30.1439°	35.4267°	77.16 %	0.00446
<b>Ma</b>	18.5177°	28.7456°	15.34 %	0.00266
<b>SG</b>	-5.4385°	-33.4956°	7.50 %	0.00134

TABLE I

RECONSTRUCTION POINTS FOR  $s = -83$ . THE LABELS OF THE RECONSTRUCTION POINTS ARE BASED ON THE INTERPRETATIONS OF THE CLUSTERS GIVEN LATER IN THIS PAPER.

In order to identify these reconstruction points we make a table of the conditional probabilities of the reconstruction points given the site.

Site	Pe	Ma	SG
Massawa Mosque	100.0	0.0	0.0
Huaisheng Mosque	86.9	0.1	13.0
Hama Great Mosque	91.1	8.9	0.0
Palmyra Congregational	92.2	7.8	0.0
Amr ibn -Al-As	100.0	0.0	0.0
Sidi Ghanem	0.0	0.0	100.0
Graveyard of Sidi 'Ukba	0.1	58.2	41.7
Qasr Humeima	100.0	0.0	0.0
Zawailah	0.0	100.0	0.0
Dome of the Chain	100.0	0.0	0.0
Ka'ba	100.0	0.0	0.0
Qasr El-Bai'j	79.2	29.6	0.0
Um Jimal Later Castellum	84.7	15.3	0.0
Kathisma Church	100.0	0.0	0.0
Qasr Mushash	86.1	13.9	0.0
Seven Sleepers Mosque	92.1	7.9	0.0
Husn Umayyad Mosque	97.4	2.6	0.0
Zeila Qiblatain Mosque (Rt)	100.0	0.0	0.0
Zeila Qiblatain Mosque (Lft)	100.0	0.0	0.0

TABLE II

SOFT CLASSIFICATION OF SITES OLDER THAN 700 CE. PROBABILITIES ARE GIVEN IN PERCENT.

First we observe that the Sidi Ghanem mosque is the only mosque that has a significant contribution to cluster SG. The mosque has been rebuilt many times and it is not clear which wall was the original qibla wall of this mosque [22]. Compared with the rest of the data set we consider Sidi Ghanem as an outlier and we remove it from the data set.

If the Sidi Ghanem mosque is removed from the data set Graveyard of Sidi 'Ukba will get about 99.9 % probability of reconstruction point Ma and 0.1 % probability of reconstruction point Pe. Thus the reconstruction point Ma essentially only have contributions from the Graveyard of Sidi 'Ukba and the Zawaila Congregational Mosque. The rest of the sites only give marginal contributions to this reconstruction point. Sidi Ghanem and these two sites all lie in Magreb, i.e. North Africa to west of Egypt. Gibson has classified all sites in his database in Magreb as having the "parallel qibla" [10]. Since there are only these three sites in the Magreb from this early period, we can only observe that their qiblas are significantly different from the qiblas from the rest of the sites. Here we will consider these three sites as outliers of the data set, i.e. they are so untypical that we will remove them from the data set and analyze the rest of the data set without them.

#### VII. DETERMINATION OF RATE AND RECONSTRUCTION POINT

When the outliers have been removed and we use  $s = -68$  as the value of the slope we get a single reconstruction point with coordinates  $30.1286^\circ N$   $35.4170^\circ E$ . The mean distortion for these early sites is 0.00481 corresponding to a circular standard deviation of  $5.6^\circ$ . If the slope is lowered  $s \leq -69$  the optimal reconstruction point starts to bifurcate into a number of optimal reconstruction points that are located very close to each other as illustrated in Table III. This is a strong indication that compression with  $s \leq -69$  leads to compressing the noise rather than the signal.

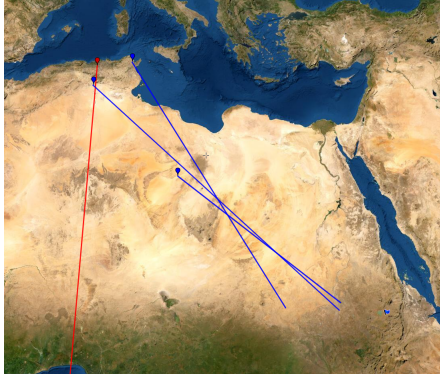


Fig. 2. The red marker and line indicate the Sidi Ghanem Mosque and its orientation. The blue markers and blue full lines indicate the locations and bearings Sidi 'Ukba Graveyard and the Zawailah mosque. The blue square is the optimal reconstruction point of cluster Ma. The reconstruction point MA is not at the intersection of the blue lines because there are also some small contributions from sites in the Levant. The blue marker with the dashed line illustrates the only mosque in Magreb between 700 CE and 750 CE.

Rec. point	Latitude	Longitude	Weight	Distortion
Pe1	30.1286°	35.4170°	99.87 %	0.00481
Pe2	30.5953°	35.3761°	0.13 %	0.00498

TABLE III

CLUSTERS AT SLOPE  $s = -69$  FOR SITES DATED BEFORE 700 CE. OUTLIERS WERE REMOVED.

### VIII. CALCULATION OF A DESCRIPTIVE CONFIDENCE REGION

The use of confidence regions is widely used in statistics, but due to the complexity of the data and the model we do not have formulas for calculating such confidence regions. Instead we will use bootstrap techniques as described in [23, Sec. 5.2] to calculate a region that resembles the well-known notion of a confidence region. The interpretation is closely related to the notion of cross validation.

The optimal reconstruction point of cluster Pe is obtained by minimizing the mean distortion where each of the 17 sites has weight 1. One may argue that a large congressional mosque should have larger weight than a small rural mosque. If a mosque has two qiblas one may ask if each of the two qiblas should have the same weight as a mosque with a single qibla. One may also ask if a mosque is rebuilt with the same qibla should count as one or two. Finally some may question the dating of some of the mosques. They may argue that these mosques should be removed from the data set. One of the main purposes in natural sciences for making controlled experiments is to obtain exchange-ability of the individual results. Exchange-ability implies that all data points should have the same weight. In humanities we often face the problem that data is not collected by controlled experiments. Therefore there is no default reason why all sites should have the same weight. Therefore we will examine what happens if we randomly assign weights to the sites before we calculate the optimal reconstruction point.

The random weights are assigned by re-sampling. From the 17 sites we sample 17 sites with replacement. In such

a bootstrap sample only about 67 % of the original sites will appear and some of the sites will appear several times such that the sites in the bootstrap sample will have multiplicities that sum to 17. This corresponds to assigning random integer weights to the sites. For this bootstrap sample we find the optimal reconstruction point where the mean distortion is calculated with weights of the sites given by the multiplicity specified by the bootstrap sample. In principle we go through all possible re-samples. Then we approximate the distribution of reconstruction points by a 2 dimensional Gaussian distribution. Finally we calculate the ellipse that contains 95 % of probability mass of the 2 dimensional Gaussian distribution and this will be our descriptive confidence region.

If the same procedure is used for a binomial distribution one will get the formula for calculation of the z-interval of the success probability.

Instead of going through all  $17^{17}$  ways of re-sampling we just randomly take 1000 re-samples and base our calculations on that. This bootstrap re-sampling is implemented by the bootstrap package in the R program. The resulting descriptive confidence region is depicted in Figure 3. The descriptive confidence region is closely confined around the ancient city of Petra. Inside the region there are no obvious alternative candidates for an early Islamic qibla.

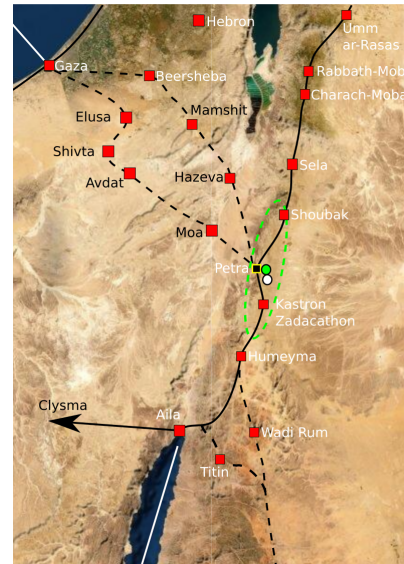


Fig. 3. The white dot is the optimal reconstruction point for cluster Pe based on 17 sites. The confidence region is outlined by the green ellipse and its center is marked by a green dot.

Jacob of Edessa stated very clearly that the mhaggriye (Arabs) pray towards the Ka'ba. Since the optimal reconstruction point is consistent with the text of Jacob of Edessa, we believe that he refers to the qibla associated with cluster Pe. There must have existed a Ka'ba in the area associated with cluster Pe before 700 CE. D. Gibson has conjectured that the bearing to Petra was used as qibla until the second fitna. Here we will make a preliminary test of this hypothesis.

As our descriptive confidence method shows the the con-

clusion will not alter if we remove a few of the sites from the data set or if we gave the sites slightly different weights. One problem about this analysis is that it may suffer from selection bias. In this period the mosques do not have mihrabs so the identification of whether a site has a qibla may have been effected by how the building is oriented compared with Petra and Mecca. For this reason we will use the later mosques to cross validate our temporary conclusion that early Muslims outside Magreb used Petra as qibla.

#### IX. CROSS VALIDATION USING LATER MOSQUES

The analysis of mosques from before 700 CE suggests that the early Muslims were able to determine the qibla with a mean distortion of 0.00481 corresponding to a circular standard deviation of  $5.6^\circ$ . Now we can compress the data involving all sites before 750 CE using the same distortion level. A sort summary of the results are as follows:

- All the mosques in the Magreb appear as outliers and are removed from the data set.
- The reconstruction point Pe associated with Petra appear again with approximately the same coordinates and the same confidence region.
- A reconstruction point Je appears with Jerusalem as qibla. Only a single mosque at Qasr Tuba is associated with this qibla with high confidence. A few other mosques could also have had this qibla.
- Two new reconstruction points emerge south and south east of Petra. If the slope is increased to  $s \geq -29$  then these two reconstruction points merge into a single cluster that we will label Ru.

One may conjecture that the early Muslims were, for some unknown reason, not able to bearing to the reconstruction point Ru with the same precision as the mosques facing Petra. The result can be seen in Table IV. With this compression the distortion of the cluster Ru is 0.01329 corresponding to a circular standard deviation of  $9.4^\circ$ . The mosques associated with the reconstruction point Ru are approximately the same as the ones that D. Gibson classified as having “the between qibla” [10]. According to Dan Gibson’s theories “the between qibla” was used because of political tensions between the ruling Umayyad dynasty and the Abbas family that would later establish the Abbasid dynasty. According to his theories the mosques that used “the between qibla” used the bisector or median between the bearing to Petra and the bearing to Mecca.

Rec. point	Latitude	Longitude	Weight	Distortion
Je	$31.7781^\circ$	$35.2353^\circ$	2.57 %	0.00662
Pe	$30.3289^\circ$	$35.4433^\circ$	34.01 %	0.01134
Ru	$27.6664^\circ$	$36.2188^\circ$	63.42 %	0.01329

TABLE IV

CLUSTERS AT SLOPE  $s = -29$  WITH OUTLIERS REMOVED AND PETRA AND JERUSALEM FIXED.

In order to calculate a confidence region for the new cluster Ru we fix the reconstruction points for the cluster Pe to Petra and fix the reconstruction point of the cluster Je to the Dome



Fig. 4. The green curve is placed around the 95 % confidence region based on a compression with  $s = -77$ . The green dot is the optimal reconstruction point of the cluster Ru. The purple marker indicates the midpoint between Petra and Mecca along a great circle.

of the Rock in Jerusalem. With these reconstruction points fixed we calculate the optimal reconstruction point of cluster Ru and calculate a confidence region around this point. The confidence region for cluster Ru is illustrated in Figure 4. The confidence region does not contain any ancient settlements and it does not intersect any of the ancient trade routes. The confidence region does not contain the midpoint between Petra and Mecca as suggested by D. Gibson. The optimal reconstruction point is located at  $27.6664^\circ E$   $36.2188^\circ N$ . The only site of archaeological relevance within the confidence region is the Ruwafa Temple that lies completely isolated in the desert [24, p. 44-49]. The temple has not been excavated but some interesting inscriptions reveal that it was built during the reign of the Roman Emperors Marcus Aurelius and Lucius Verus. The observation that the Ruwafa Temple is almost at the optimal reconstruction point is new knowledge and it is still far from clear how this result should be interpreted.

We can test Gibson’s “between qibla” versus our “Ruwafa qibla” in the same way as we tested great circles versus rhumb lines and the result is that “Ruwafa qibla” gives a much better description of the data than the “between qibla”. It is surprising that none of our reconstruction points lie near Mecca. For instance there is an rock inscription that testify that Mashid al-Haram was built (or rebuilt) in Mecca in 698 CE [25, p. 111], and there are some mosques before the Abbaside revolution that seem to face Mecca, but they are so few that they do not form a cluster.

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